Algebra, Number Theory and Combinatorics

Individual

Problem 1. Let G be a finite group, $H \leq G$ a subgroup, $N = N_G(H)$ its normalizer and $N' = N_G(N)$ the normalizer of its normalizer. Assume that the order $|H|$ of H is coprime to the index $[N : H]$. Prove that $N = N'$.

Problem 2. Let F be a field. A Zariski closed subset Y of $\bar{F}^n$ is F-*closed* if the ideal I corresponding to Y is defined over F. A subset of $\bar{F}^n$ is F-*open* if its complement is F-closed. Do the F-open sets always form a topology on $\bar{F}^n$?

Problem 3. (a) Let A be the $\mathbb{C}$-algebra generated by the symbols a_i, b_i, $i = 1, ..., n$, with the relations

$$a_i b_j + b_j a_i = \delta_{ij} 1.$$

Classify all simple A-modules.

(b) What if the relations are replaced by these

$$a_i b_j - b_j a_i = \delta_{ij} 1 \ ?$$

Problem 4. Let $A = \mathbb{C}[t, t^{-1}]$. Consider the following endomorphism γ of A

$$\gamma : A \longrightarrow A, \quad \sum_m a_m t^m \mapsto \sum_m \overline{a_m} \left(\frac{-1}{t} \right)^m .$$

Compute explicitly $R := \{f \in A | \gamma(f) = f\}$ and show that R is finitely generated as an $\mathbb{R}$-algebra. Is R a UFD or even a PID?